Google DeepMind

Equilibrium-Invariant Embedding, Metric Space, and Fundamental Set of 2×2 Normal-Form Games

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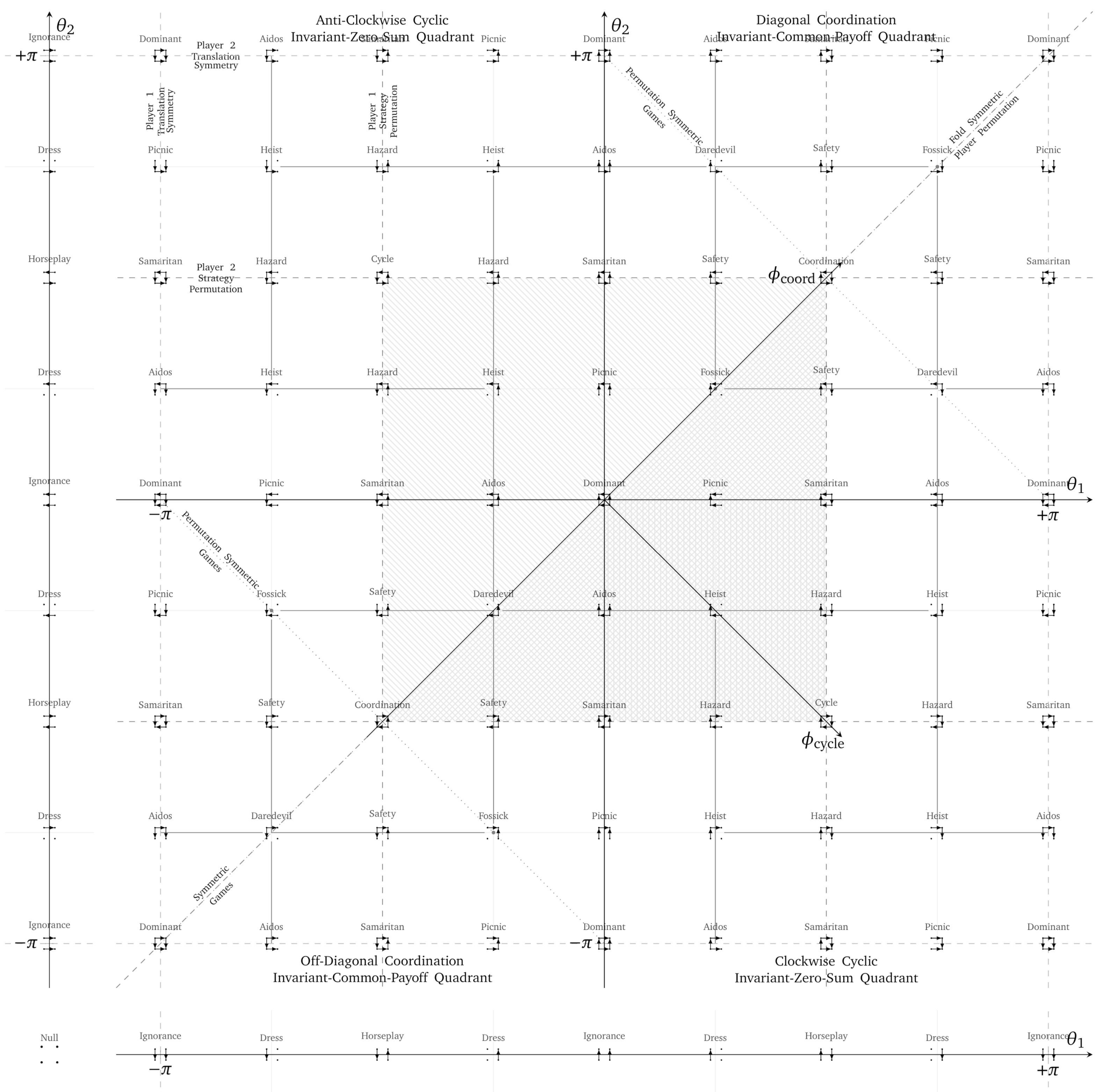
Introduction

- Equilibria solution concepts, like Nash equilibrium, correlated equilibrium, and coarse correlated equilibrium, are important predictors of rational behaviour in normal-form games.
- Certain payoff transforms to normal-form games either do not change the set of equilibria (e.g. offset invariances), or result in simple analogous transforms to the set of equilibria (e.g. symmetries).
- These transforms can be used to reduce the degrees of freedom or volume of the space of games.

Contributions

- Proves an upper bound on the number of degrees of freedom ($NM^{(N-1)} + N$) that can be linearly removed from a normal-form game.
- Identifies transforms that fully remove this number of degrees of freedom (other player strategy offset and per-player payoff scaling).
- Defines an equilibrium-invariant embedding and distance metric between normal-form games.
- Notes that only two variables (a reduction from eight) are needed to parameterize all 2×2 games without losing any equilibrium information.
- Proposes a novel visualization of the space of equilibrium-invariant embeddings of 2×2 games. This visualization captures interesting properties including: equilibrium support, cycles, competition, coordination, distances, best-responses, and symmetries. If significantly simpler and more structured than other proposed topologies.
- Discovers an additional better-response-invariant transform. This transform can be used to rediscover a set of 15 fundamental 2×2 games.
- Defines names and a graphical visualization for each game in the fundamental set.
- Proposes a method for visualizing the strategic space for larger extensive-form games.

2×2 Equilibrium-Invariant Embedding



Equilibrium-Invariant Transform

Affine transform:

$$G_p(a) \to \hat{G}_p(a) = s_p G_p(a_p, a_{-p}) + b_p(a_{-p})$$

CCE Deviation Gains:

$$A_p^{\text{CCE}}(a'_p, a) := G_p(a'_p, a_{-p}) - G_p(a)$$

Sketch Proof:

$$A_p^{\text{CCE}}(a'_p, a) \to s_p G_p(a'_p, a_{-p}) + b_p(a_{-p}) - s_p G_p(a) - b_p(a_{-p})$$

Embedding and Distance Metric

Embedding:

$$G_p^{equil}(a) = \frac{1}{Z}\left(G_p(a) - \frac{1}{|\mathcal{A}_p|}\sum_{a_p} G_p(a_p, a_{-p})\right)$$

$$Z = \left\| G_p - \frac{1}{|\mathcal{A}_p|}\sum_{a_p} G_p(a_p, a_{-p}) \right\|_F$$

Distance Metric:

$$d^{equil}\left((G^A, \epsilon^A), (G^B, \epsilon^B)\right) = \sqrt{\sum_p \arccos\left(\sum_a G_p^{A,equil}(a) G_p^{B,equil}(a)\right)^2}$$

2×2 Equilibrium-Invariant Embedding

- Only requires two variables.
- Derived from the affine transform.
- All equilibria are preserved.
- Geometrically, a circle per player.

$$G_1 = \begin{bmatrix} g_1^{AA} & g_1^{AB} \\ g_1^{BA} & g_1^{BB} \end{bmatrix} \to G_1^{equil}(\theta_1) = \begin{bmatrix} \frac{1}{\sqrt{2}}\sin(\theta_1 + \frac{\pi}{4}) & \frac{1}{\sqrt{2}}\cos(\theta_1 + \frac{\pi}{4}) \\ -\frac{1}{\sqrt{2}}\sin(\theta_1 + \frac{\pi}{4}) & -\frac{1}{\sqrt{2}}\cos(\theta_1 + \frac{\pi}{4}) \end{bmatrix}$$

$$G_2 = \begin{bmatrix} g_2^{AA} & g_2^{AB} \\ g_2^{BA} & g_2^{BB} \end{bmatrix} \to G_2^{equil}(\theta_2) = \begin{bmatrix} \frac{1}{\sqrt{2}}\sin(\theta_2 + \frac{\pi}{4}) & -\frac{1}{\sqrt{2}}\sin(\theta_2 + \frac{\pi}{4}) \\ \frac{1}{\sqrt{2}}\cos(\theta_2 + \frac{\pi}{4}) & -\frac{1}{\sqrt{2}}\cos(\theta_2 + \frac{\pi}{4}) \end{bmatrix}$$

Visualising Larger Extensive-Form Games

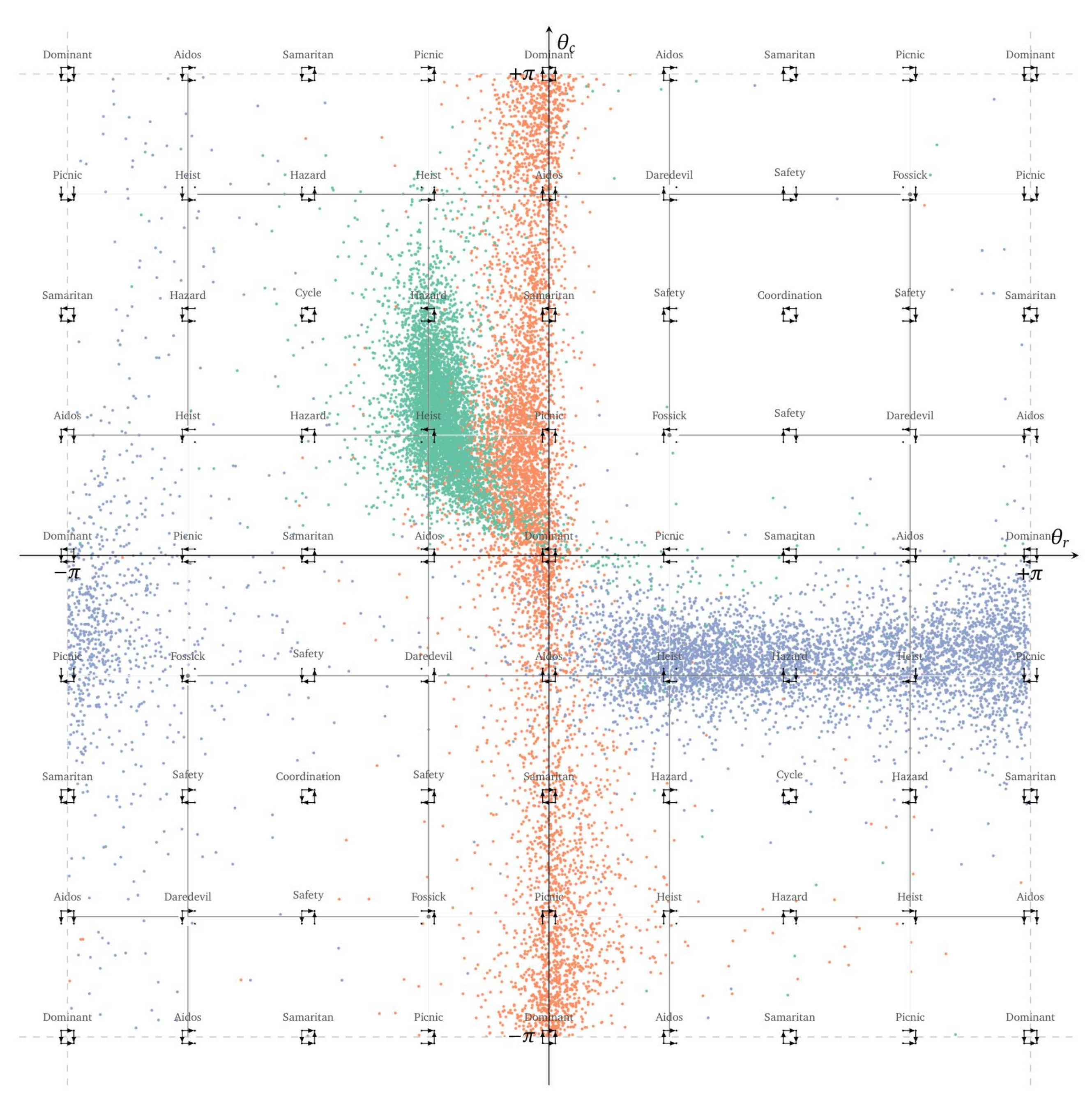
2×2 normal-form games can be described by their two embedding parameters.

Larger two-player normal-form games can be described by a point cloud of every possible 2×2 subgame.

N-player games can be approximated around a joint strategy, considering their local deviations (similar to a polymatrix approximation).

Extensive-form games can be approximated by sampling policies (rather than considering every deterministic strategy).

Right Figure: A local polymatrix approximation visualization of three-player Leduc poker around the background strategies: always raise, always call, and always fold.



■ raise vs call, ■ call vs fold, ■ fold vs raise